How a single-factor CAPM works in a multi-currency world: Results from the latest research

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ABSTRACT

At the Actuarial Society's convention in 2012, the authors presented a paper entitled "How a single-factor CAPM works in a multi-currency world". Some problems have since been found with the application of the methodology of that paper. In this paper the single-factor multi-currency capital-asset pricing model developed in the earlier paper is revised. A new approach is adopted, which resolves those problems.

As stated in the previous paper, the advantage of using a single-factor model is that it does not treat currency risks as carrying different weight from investment risks; regardless of its source, risk is measured as variance and weighted accordingly. The aim of this research is primarily to give actuaries a way ahead in the use of the single-factor CAPM in a multi-currency world for the purposes of the stochastic modelling of the assets and liabilities of long-term financial institutions such as pension funds, particularly for the purposes of liability-driven investments and market-consistent valuation, and the application of the model has been designed with that intention. However, it is envisaged that the model will also be of interest to other practitioners.

The authors' major original contribution to the literature is their proof that, for a single-factor CAPM to work in a multi-currency world, there is a necessary condition. Because of the revision of their approach, it has been necessary to restate that condition. As before, the theory is applied to two major currencies and two minor currencies, namely the USA dollar, the UK pound, the South African rand and the Turkish lira.

KEYWORDS

International CAPM, single-factor multi-currency CAPM (SFM-CAPM)

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1. INTRODUCTION

1.1 In an earlier paper (Thomson, Şahin & Reddy, unpublished) the authors presented a method of applying the capital-asset pricing model (CAPM) in a multi-currency world. Some problems have since been found with the application of the method adopted in that paper. Specifically, the problem was that, depending on the number of currencies and the number of assets issued in each currency, the number of constraints might be greater than the number of equations to be solved in order to determine the optimal portfolio. In this paper the single-factor multi-currency CAPM (SFM-CAPM) developed in Thomson, Şahin & Reddy (op. cit.) is revised. A new approach is adopted, which resolves those problems.

1.2 The advantage of using a single-factor model is that it does not treat currency risks as carrying different weight from investment risks; regardless of its source, risk is measured by an investor as the variance of the return in the currency in which that investor measures risk, and weighted accordingly. This matter is further discussed in Thomson, Şahin & Reddy (op. cit.). Unlike international CAPMs developed in the literature to date, and unlike the notion of a unique single-factor CAPM across all currencies (the notion rejected by Wilkie (unpublished)), this paper assumes, as in Thomson, Şahin & Reddy (op. cit.) that, for every currency in which investors measure risk, there is a CAPM that is unique to those investors across all the markets in which they invest, but different from the CAPM of investors who measure their risks in other currencies. It also assumes that, regardless of the currency in which they measure risk, all investors have homogeneous expectations and all investors participate in the formation of equilibrium. It develops a theory for multi-currency CAPMs by developing a CAPM for each set of investors that measure their returns in a particular currency. In the development of this theory the meanings of 'homogeneous expectations' and of 'equilibrium' are reconsidered in the context of a multi-currency world.

1.3 Literature on the domestic CAPM, market segregation, market integration and the international versions of the CAPM is reviewed in Thomson, Şahin & Reddy (op. cit.). To avoid unnecessary repetition that review is not repeated here. In section 2 the SFM-CAPM is developed and the necessary condition for the SFM-CAPM is derived.

In order to resolve the problem referred to above, that condition is implemented by means of a penalty method in the specification of an objective function for the estimation of *ex-ante* expected returns. The method is applied to two major currencies and two minor currencies, namely the USA dollar, the UK pound, the South African rand and the Turkish lira. The data obtained for this purpose are described in Thomson, Şahin & Reddy (op. cit.) and the results of the application are described in section 3. The application illustrated in this paper is designed for use by actuaries in the modelling of the assets and liabilities of long-term financial institutions. To that end, the longest possible range of time periods is used and quarterly intervals are used rather than the relatively short time intervals typically used in the literature. However, the model could be applied to shorter periods and shorter time intervals. Section 4 concludes the paper with a summary of the findings and some indications of the way in which those findings will lead to further research for the purposes of such modelling.

2. THE NECESSARY CONDITION FOR THE SINGLE-FACTOR MULTI-CURRENCY CAPM

In this section an SFM-CAPM is developed. Section 2.1 is a preliminary discussion largely devoted to the definitions required for the following sections. Much of that section follows Thomson, Şahin & Reddy (op. cit.); it is repeated here for convenience. Some of the notation in that section has been simplified. In section 2.2 it is shown that, for a single-factor CAPM to work in a multi-currency world, there is a necessary condition. That section has been revised. The reason for this is that, because we are now using a penalty method, the necessary condition does not always apply exactly. In Thomson, Şahin & Reddy (op. cit.) the SFM-CAPM constraint was defined with reference to the numeraire currency. Whilst that was satisfactory when the constraints were intended to be exact, it is not satisfactory when the constraints are treated as approximations to be minimised. This is because, in the latter, undue weight would be applied to the numeraire currency and the results would depend on the choice of numeraire. The SFM-CAPM is formulated in section 2.3. As explained above, that formulation is expressed in terms of a penalty function, which requires an optimisation process. Problems relating to local optima are discussed in section 2.4. Section 2.5 considers a special case.

2.1 Preliminary Discussion

2.1.1 Suppose there are C currencies and that, in currency c, there is one risk-free asset and n_c risky capital assets have been issued. It is assumed that every investor measures investment returns in one of these currencies. Regardless of the currency in which an investor measures investment returns, the investor may invest in any currency. An 'asset issued in currency c' is a risky asset issued in that currency or the risk-free asset denominated in that currency. (For an investor who measures returns in another currency, the risk-free asset denominated in currency c is not risk-free; this matter is dealt with in greater detail below.)

2.1.2 The 'return in currency *c*' on an asset issued in currency *d* is the force of return (i.e. the 'log return') earned on that asset, over a unit interval, measured in currency *c*. Thus, for example, if the value in currency *d* of asset *i* issued in that currency changes over a unit interval from Y_{di0} to Y_{di1} then the return on that asset during that interval, measured in that currency, is:

$$X_{di} = \ln \frac{Y_{di1}}{Y_{di0}}.$$

If during that interval the exchange rate between currency *d* and a numeraire currency—say currency 1—changes from Y_{d0} units of currency 1 per unit of currency *d* to Y_{d1} such units then the increase in the exchange rate is:

$$X_d = \ln \frac{Y_{d1}}{Y_{d0}}.$$

2.1.3 X_d is thus a measure of the strengthening of currency *d* against currency 1 or, where it is negative, of the weakening of that currency. The value of the asset issued in currency *d*, measured in currency 1, changes from $Y_{di0}Y_{d0}$ to $Y_{di1}Y_{d1}$ and the value of that asset, measured in currency *c*, changes from $\frac{Y_{di0}Y_{d0}}{Y_{c0}}$ to $\frac{Y_{di1}Y_{d1}}{Y_{c1}}$. The return on that asset during that interval, measured in currency *c*, is:

$$\ln \frac{Y_{di1}Y_{d1}Y_{c0}}{Y_{di0}Y_{d0}Y_{c1}} = X_{di} + X_d - X_c.$$

2.1.4 Where exchange rates are expressed in units of currency *d* per unit of currency 1 (as where currency 1 is the US dollar), it should be noted that the exchange rate should be inverted so as to measure the strength of currency *d* against currency 1.

2.1.5 Exchange rates are measured per unit of currency 1. The rate of strengthening of currency *c* per unit of currency 1 is measured as a force over the unit interval, thus avoiding the need for compounding that would otherwise apply. Returns and rates of strengthening of currencies may be measured in real terms (relative to a price index) or in nominal terms. Whilst practitioners commonly record and report returns in nominal terms, it may be preferable to do so in real terms. For this reason the authors have explored the use of real returns as well as nominal returns. Again, the use of forces avoids the compounding effects of inflation.

2.1.6 We assume that the CAPM applies for investors in each currency. More specifically, we assume that:

1) investors who measure their investment returns in currency *c* (i.e. 'currency-*c* investors') have indifference curves in mean–variance space, the means and variances being those measured in that currency; and

2) all investors, regardless of the currency in which they measure returns, have homogeneous expectations of the means, variances and covariances of:

(a) the returns in each currency on assets issued in that currency; and

(b) rates of strengthening of each currency.

2.1.7 First we consider returns in currency c on assets issued in that currency. For this purpose we define the following random variables, where, for $c=1,\dots,C$, $i=1$ denotes the risk-free asset and $i=2,\dots,n_c$ the risky assets issued in that currency:

— X_{ci} is the return in currency c on asset i issued in that currency for $c=1,\dots,C$; $i=1,\dots,n_c$; and

— X_c is the rate of strengthening of currency c for $c=2,\dots,C$.

2.1.8 Here n_c is the number of assets issued in currency c. For $i=1$ $X_{c1}=r_c$, which is deterministic, being the return on the risk-free asset denominated in currency c. For $i=2,\dots,n_c$ X_{ci} is a random variable.

2.1.9 We define the following parameters, where, as above, for $c=1,\dots,C$, $i=1$ denotes the risk-free asset denominated in that currency and $i=2,\dots,n_c$ denotes the risky assets issued in that currency:

— μ_{ci} is the expected return in currency c on risky asset i issued in that currency; i.e.: $\mu_{ci}=E\{X_{ci}\}$;

— $\sigma_{ci,dj}$ is the covariance of the return in currency c on risky asset i issued in that currency with the return in currency d on risky asset j issued in that currency;

i.e.: $$\sigma_{ci,dj}=\begin{cases}\operatorname{var}\{X_{ci}\} & \text{for } d=c, j=i;\\ \operatorname{cov}\{X_{ci},X_{dj}\} & \text{otherwise;}\end{cases}$$

— μ_c is the expected rate of strengthening of currency c; i.e.: $\mu_c=E\{X_c\}$;

— $\sigma_{c,di}$ is the covariance of the rate of strengthening of currency c with the return in currency d on risky asset i issued in that currency; i.e.: $\sigma_{c,di}=\operatorname{cov}\{X_c,X_{di}\}$;

— $\sigma_{c,c}$ is the variance of the rate of strengthening of currency c; i.e.: $\sigma_{c,c}=\operatorname{var}\{X_c\}$.

2.1.10 Because investors have homogeneous expectations (assumption (2) above), the means, variances and covariances defined above are the same for all investors, regardless of the currency in which they measure their returns. Because the expected values and the variances and covariances are those of forces rather than rates of return, the values will be different from those typically used. Whilst for the standard CAPM mean–variance analysis is expressed in terms of rates of return, here it is expressed in terms of forces. Utility functions—and therefore indifference curves—may similarly be expressed in terms of forces of return.

2.1.11 In the case of currency 1 the rate of strengthening is trivially zero. For that currency we therefore have:

$$\mu_1 = 0; \tag{1}$$

$$\sigma_{1,di} = 0; \text{ and} \tag{2}$$

$$\sigma_{1,1} = 0. \tag{3}$$

2.1.12 Also, for the risk-free asset denominated in currency c, the return is deterministic, so we have:

$$\sigma_{c1,dj} = 0. \tag{4}$$

2.1.13 The variables defined above relate to returns in a particular currency as measured in that currency and to exchange rates between that currency and currency 1. Now we need to consider the returns to investors who measure their returns in other currencies, for example a currency-c investor. For this purpose we define the following:

— X_{di}^c is the return in currency c on asset i issued in currency d for $c,d=1,\ldots,C$; $i=1,\ldots,n_d$; i.e.:

$$X_{di}^c = X_{di} + X_d - X_c. \tag{5}$$

In particular:

$$X_{di}^d = X_{di}.$$

2.1.14 Note that subscripts are used to denote the currency in which an asset is issued and the category of that asset, whereas superscripts are used to denote the currency in which an investor measures returns; the former relates to the asset, whereas the latter relates to the investor.

2.1.15 Because we are working with forces of strengthening of currencies, the increases are additive. We may then determine the following:

— μ_{di}^c is the expected return in currency c on asset i issued in currency d for $c,d=1,\ldots,C$; $i=1,\ldots,n_d$; i.e.:

$$\mu_{di}^c = E\left\{X_{di} + X_d - X_c\right\} = \mu_{di} + \mu_d - \mu_c. \tag{6}$$

— $\sigma_{di,ej}^c$ is the covariance of the return in currency c on asset i issued in currency d with the return in currency c on risky asset j issued in currency e; i.e., from equation (5):

$$\begin{aligned}\sigma^c_{di,ej} &= \operatorname{cov}\left\{X_{di}+X_d-X_c, X_{ej}+X_e-X_c\right\}\\ &= \operatorname{cov}\left\{X_{di,}X_{ej}\right\}+\operatorname{cov}\left\{X_{di},X_e\right\}-\operatorname{cov}\left\{X_{di},X_c\right\}\\ &\quad+\operatorname{cov}\left\{X_dX_{ej}\right\}+\operatorname{cov}\left\{X_d,X_e\right\}-\operatorname{cov}\left\{X_dX_c\right\}\\ &\quad-\operatorname{cov}\left\{X_cX_{ej}\right\}-\operatorname{cov}\left\{X_cX_e\right\}+\operatorname{var}\left\{X_c\right\}\\ &= \sigma_{di,ej}+\sigma_{e,di}-\sigma_{c,di}+\sigma_{d,ej}+\sigma_{d,e}-\sigma_{c,d}-\sigma_{c,ej}-\sigma_{c,e}+\sigma_{c,c}.\end{aligned} \tag{7}$$

2.1.16 We refer to μ_{di} and μ_d as the 'underlying expectations' and to μ^c_{di} as the 'expected returns to investors'.

2.1.17 Equation (7) is required in order to determine the variance of the return on the portfolio of risky assets to a currency-c investor as explained below (cf. equation (15)).

2.1.18 Let p^c_{di} denote the value in currency c of investments in asset i issued in currency d held by currency-c investors, per unit of the total value in that currency of the assets held by such investors. The value of p^c_{di} is unknown; it is estimated through an optimisation process explained below. We now define the portfolio of risky assets held by a currency-c investor as:

$$\left\{p^c_{di} \mid (d,i)\in\Psi_c\right\} \tag{8}$$

where:

$$\Psi_c = \left\{(d,i) \mid d\in\{1,\ldots,C\}; i\in\Omega^c_d\right\}; \text{ and}$$

$$\Omega^c_d = \begin{cases}\{2,\ldots,n_d\} & \text{for } d=c;\\ \{1,\ldots,n_d\} & \text{for } d\neq c;\end{cases}.$$

2.1.19 The set Ω^c_d has n_d+1 elements for $d\neq c$ or n_c for $d=c$. This is because, for currency $d\neq c$, the risk-free asset denominated in currency d is included (as p^c_{d1}) as a risky asset in this portfolio, whereas for currency $d=c$, the risk-free asset denominated in that currency is not included, as it is not a risky asset. By definition, the elements of the set $\left\{p^c_{di} \mid (d,i)\in\Psi_c\right\}$ sum to 1; i.e.:

$$\sum_{(d,i)\in\Psi_c} p^c_{di} = 1. \tag{9}$$

2.1.20 Similarly, we define the returns on the risky assets held in currency d by a currency-c investor as:

$$\left\{X^c_{di} \mid (d,i)\in\Psi_c\right\}; \tag{10}$$

where X_{di}^{c} is the return on risky asset *i* issued in currency *d* measured in currency *c* (equation (5)).

2.1.21 We similarly define the expected return on the risky assets held in currency *d* by a currency-*c* investor as:

$$\left\{\mu_{di}^{c} \mid (d,i) \in \Psi_{c}\right\}; \tag{11}$$

where μ_{di}^{c} is the expected return on risky asset *i* issued in currency *d* measured in currency *c* (equation (6)).

2.1.22 Also, we define the covariances of the returns on the risky assets held in currency *d* with those on the risky assets held in currency *e* by a currency-*c* investor as:

$$\left\{\sigma_{di,ej}^{c} \mid (d,i),(e,j) \in \Psi_{c}\right\}; \tag{12}$$

where $\sigma_{di,ej}^{c}$ is the covariance of the returns on risky assets *i* and *j* held in currencies *d* and *e* respectively by a currency-*c* investor (equation (7)).

2.1.23 Now, from the definitions in equations (8) and (10), we may express the return on the portfolio of risky assets held by a currency-*c* investor (i.e. on the 'market portfolio' of currency-*c* investors) as:

$$X_{\mathrm{M}}^{c} = \sum_{(d,i)\in\Psi_{c}} p_{di}^{c} X_{di}^{c}\,. \tag{13}$$

(We use the subscript M to denote that portfolio.) Similarly, from the definitions in equations (8) and (11), we may express the expected return on the portfolio of risky assets held by a currency-*c* investor as:

$$\mu_{\mathrm{M}}^{c} = E\left\{X_{\mathrm{M}}^{c}\right\} = \sum_{(d,i)\in\Psi_{c}} p_{di}^{c} \mu_{di}^{c}\,. \tag{14}$$

2.1.24 Also, from equations (8) and (12), we may express the variance of the return on the portfolio of risky assets held by a currency-*c* investor as:

$$\sigma_{\mathrm{M,M}}^{c} = \mathrm{var}\left\{X_{\mathrm{M}}^{c}\right\} = \sum_{(d,i),(e,j)\in\Psi_{c}} p_{di}^{c} p_{ej}^{c} \sigma_{di,ej}^{c}\,. \tag{15}$$

2.1.25 In terms of the CAPM, currency-*c* investors determine their portfolio of risky assets by maximising

$$k = \frac{\hat{\mu}_{\mathrm{M}}^{c} - r_{c}}{\sqrt{\hat{\sigma}_{\mathrm{M,M}}^{c}}}; \tag{16}$$

where:

$\hat{\mu}_{\mathrm{M}}^{c}$ is the *ex-ante* estimate of the expected return to a currency-c investor on her/his portfolio;

r_c is the return on the risk-free asset issued in currency c; and

$\hat{\sigma}_{\mathrm{M,M}}^{c}$ is the *ex-ante* estimate of the variance of the return to a currency-c investor on her/his portfolio.

2.1.26 In practice the value of r_c will be known. For the purposes of this paper a neutral value was used, determined as the sample mean of the return on the risk-free asset in currency c:

$$r_c = \hat{\mu}_{c1}^{c} = \hat{\mu}_{c1}. \tag{17}$$

2.1.27 In order to avoid short positions in the market portfolio of currency-c investors, k is maximised subject to the constraints:

$$p_{di}^{c} \geq 0 \text{ for all } (d,i) \in \Psi_c \text{ and for all } c;$$

and, as in equation (9):

$$\sum_{(d,i)\in\Psi_c} p_{di}^{c} = 1.$$

In practice it may be appropriate to apply other constraints, particularly constraints on investments abroad. Those constraints may give more meaningful results than the simple constraint imposed above.

2.1.28 This gives the tangency portfolio, i.e. the portfolio on the efficient frontier in mean–standard-deviation space at which the straight line intersecting the mean axis at r_c is tangential to the efficient frontier. Under the CAPM, the latter line is the capital-market line and the tangency portfolio is the market portfolio. k, the Sharpe ratio, is the slope of the capital-market line.

2.2 A Necessary Condition

Let $\sigma_{di,\mathrm{M}}^{c}$ be the covariance of the return in currency c on asset i issued in currency d with the return in currency c on the market portfolio of a currency-c investor, i.e.:

$$\sigma_{di,\mathrm{M}}^{c} = \mathrm{cov}\left\{X_{di}^{c}, X_{\mathrm{M}}^{c}\right\} = \sum_{(e,j)\in\Psi_c} p_{ej}^{c}\sigma_{di,ej}^{c}\,; \tag{18}$$

and, as in equation (15), let $\sigma_{\mathrm{M,M}}^{c}$ be the variance of the return in currency c on the market portfolio of a currency-c investor, i.e.:

$$\sigma_{\mathrm{M,M}}^{c} = \mathrm{var}\left\{X_{\mathrm{M}}^{c}\right\} = \sum_{(d,i),(e,j)\in\Psi_c} p_{di}^{c}p_{ej}^{c}\sigma_{di,ej}^{c}\,. \tag{19}$$

Now let:

$$\kappa_{di}^{c}=\frac{\sigma_{di,\mathrm{M}}^{c}-\sigma_{d1,\mathrm{M}}^{c}}{\sigma_{\mathrm{M,M}}^{c}}\left(\mu_{\mathrm{M}}^{c}-r_{c}\right); \tag{20}$$

where r_c is the risk-free rate in currency c.

Theorem *If the SFM-CAPM applies in a multi-currency world then, for any currencies c and e:*

$$\kappa_{di}^{c}=\kappa_{di}^{e}\,. \tag{21}$$

Proof

Since the CAPM applies for investors in each currency (assumption (1)), it follows that, for asset *i* issued in currency *d*, the expected return in currency *c* is:

$$\mu_{di}^{c}=r_{c}+\frac{\sigma_{di,\mathrm{M}}^{c}}{\sigma_{\mathrm{M,M}}^{c}}\left(\mu_{\mathrm{M}}^{c}-r_{c}\right). \tag{22}$$

Similarly:

$$\mu_{di}^{e}=r_{e}+\frac{\sigma_{di,\mathrm{M}}^{e}}{\sigma_{\mathrm{M,M}}^{e}}\left(\mu_{\mathrm{M}}^{e}-r_{e}\right). \tag{23}$$

From equation (6) we have:

$$\mu_{di}^{c}=\mu_{di}+\mu_{d}-\mu_{c}\,. \tag{24}$$

Similarly:

$$\mu_{di}^{e}=\mu_{di}+\mu_{d}-\mu_{e}\,. \tag{25}$$

Making μ_{di} the subject of equation (25) we have:

$$\mu_{di}=\mu_{di}^{e}-\mu_{d}+\mu_{e}\,. \tag{26}$$

Substituting equation (26) into equation (24) we obtain:

$$\mu_{di}^{c}=\left(\mu_{di}^{e}-\mu_{d}+\mu_{e}\right)+\mu_{d}-\mu_{c}\,;$$

i.e.:

$$\mu_{di}^{c}+\mu_{c}=\mu_{di}^{e}+\mu_{e}\,. \tag{27}$$

Now we substitute equations (22) and (23) into equation (27) to give:

$$\left\{r_{c}+\frac{\sigma_{di,\mathrm{M}}^{c}}{\sigma_{\mathrm{M,M}}^{c}}\left(\mu_{\mathrm{M}}^{c}-r_{c}\right)\right\}+\mu_{c}=\left\{r_{e}+\frac{\sigma_{di,\mathrm{M}}^{e}}{\sigma_{\mathrm{M,M}}^{e}}\left(\mu_{\mathrm{M}}^{e}-r_{e}\right)\right\}+\mu_{e}\,;$$

i.e.:

$$r_{c}+\mu_{c}+\frac{\sigma_{di,\mathrm{M}}^{c}}{\sigma_{\mathrm{M,M}}^{c}}\left(\mu_{\mathrm{M}}^{c}-r_{c}\right)=r_{e}+\mu_{e}+\frac{\sigma_{di,\mathrm{M}}^{e}}{\sigma_{\mathrm{M,M}}^{e}}\left(\mu_{\mathrm{M}}^{e}-r_{e}\right). \tag{28}$$

From equations (24) and (25) we have, for $i = 1$:

$$\mu_{d1}^{c} = r_d + \mu_d - \mu_c \text{; and} \tag{29}$$

$$\mu_{d1}^{e} = r_d + \mu_d - \mu_e . \tag{30}$$

And from equations (22) and (23) we have, for $i = 1$:

$$\mu_{d1}^{c} = r_c + \frac{\sigma_{d1,\mathrm{M}}^{c}}{\sigma_{\mathrm{M,M}}^{c}}\left(\mu_{\mathrm{M}}^{c} - r_c\right) \text{ ; and} \tag{31}$$

$$\mu_{d1}^{e} = r_e + \frac{\sigma_{d1,\mathrm{M}}^{e}}{\sigma_{\mathrm{M,M}}^{e}}\left(\mu_{\mathrm{M}}^{e} - r_e\right). \tag{32}$$

From equations (29) and (31) we have:

$$r_d + \mu_d - \mu_c = r_c + \frac{\sigma_{d1,\mathrm{M}}^{c}}{\sigma_{\mathrm{M,M}}^{c}}\left(\mu_{\mathrm{M}}^{c} - r_c\right);$$

i.e.:

$$r_c + \mu_c = r_d + \mu_d - \frac{\sigma_{d1,\mathrm{M}}^{c}}{\sigma_{\mathrm{M,M}}^{c}}\left(\mu_{\mathrm{M}}^{c} - r_c\right). \tag{33}$$

and similarly from equations (30) and (32) we have:

$$r_e + \mu_e = r_d + \mu_d - \frac{\sigma_{d1,\mathrm{M}}^{e}}{\sigma_{\mathrm{M,M}}^{e}}\left(\mu_{\mathrm{M}}^{e} - r_e\right). \tag{34}$$

Substituting (36) and (37) into (31) we obtain:

$$\begin{aligned}&\left\{r_d + \mu_d - \frac{\sigma_{d1,\mathrm{M}}^{c}}{\sigma_{\mathrm{M,M}}^{c}}\left(\mu_{\mathrm{M}}^{c} - r_c\right)\right\} + \frac{\sigma_{di,\mathrm{M}}^{c}}{\sigma_{\mathrm{M,M}}^{c}}\left(\mu_{\mathrm{M}}^{c} - r_c\right)\\ &= \left\{r_d + \mu_d - \frac{\sigma_{d1,\mathrm{M}}^{e}}{\sigma_{\mathrm{M,M}}^{e}}\left(\mu_{\mathrm{M}}^{e} - r_e\right)\right\} + \frac{\sigma_{di,\mathrm{M}}^{e}}{\sigma_{\mathrm{M,M}}^{e}}\left(\mu_{\mathrm{M}}^{e} - r_e\right);\end{aligned}$$

i.e.:

$$\kappa_{di}^{c} = \frac{\sigma_{di,\mathrm{M}}^{c} - \sigma_{d1,\mathrm{M}}^{c}}{\sigma_{\mathrm{M,M}}^{c}}\left(\mu_{\mathrm{M}}^{c} - r_c\right) = \frac{\sigma_{di,\mathrm{M}}^{e} - \sigma_{d1,\mathrm{M}}^{e}}{\sigma_{\mathrm{M,M}}^{e}}\left(\mu_{\mathrm{M}}^{e} - r_e\right) = \kappa_{di}^{e} . \tag{35}$$

From equation (29) it may be noted that the differential between the expected returns to a currency-c investor on the risk-free assets in currencies d and f is equal to the interest-rate differential between those assets; i.e.:

$$\mu_{d1}^{c}-\mu_{f1}^{c}=\left(r_d+\mu_d-\mu_c\right)-\left(r_f+\mu_f-\mu_c\right);$$

so that:

$$\left(\mu_{d1}^{c}-\mu_d+\mu_c\right)-\left(\mu_{f1}^{c}-\mu_f+\mu_c\right)=r_d-r_f\,.$$

2.3 Formulation of the SFM-CAPM

2.3.1 Following equation (22), subject to the requirement of the Theorem, we may define both the GCAPM and the SFM-CAPM as:

$$X_{di}^{c}=r_c+\beta_{di}^{c}\left(X_{\mathrm{M}}^{c}-r_c\right)+\varepsilon_{di}^{c}\,; \tag{35A}$$

where:

$$\beta_{di}^{c}=\frac{\sigma_{di,\mathrm{M}}^{c}}{\sigma_{\mathrm{M,M}}^{c}}\text{; and}$$

$$E\left\{\varepsilon_{di}^{c}\right\}=0\,.$$

i.e., as stated in equation (22):

$$\mu_{di}^{c}=r_c+\beta_{di}^{c}\left(\mu_{\mathrm{M}}^{c}-r_c\right). \tag{35B}$$

The difference between the GCAPM and the SFM-CAPM is that, in the former, no constraints are applied in determining the values, whereas in the latter, constraints are applied. In the SFM-CAPM it is thus implicitly assumed that some of the sample values may be biased estimates of the *ex-ante* parameters required.

2.3.2 Suppose that the sample values $\hat{\sigma}_{ci,dj}$ and $\hat{\sigma}_{c,di}$ are unbiased estimates of the *ex-ante* values of $\sigma_{ci,dj}$ and $\sigma_{c,di}$, both for the GCAPM and for the SFM-CAPM. For the GCAPM, suppose that the sample values $\hat{\mu}_{ci}$ and $\hat{\mu}_{c}$ are unbiased estimates of the *ex-ante* underlying expectations, but that for the SFM-CAPM they are not. Let $\hat{\mu}_{ci}^{(\mathrm{G})}$ and $\hat{\mu}_{c}^{(\mathrm{G})}$ denote the sample values of the underlying expectations. Let $\mu_{ci}^{(\mathrm{S})}$ and $\mu_{c}^{(\mathrm{S})}$ denote the *ex-ante* values of the underlying expectations on the SFM-CAPM.

2.3.3 In principle we could determine $\hat{\mu}_{ci}^{(\mathrm{S})}$ and $\hat{\mu}_{c}^{(\mathrm{S})}$ so as to minimise:

$$D_{\mu}^{2}=\frac{1}{Q_{\mu}}\left[\sum_{c=1}^{C}\left\{\sum_{i=2}^{n_c}\left(\hat{\mu}_{ci}^{(\mathrm{S})}-\hat{\mu}_{ci}^{(\mathrm{G})}\right)^2\right\}+\sum_{c=2}^{C}\left(\hat{\mu}_{c}^{(\mathrm{S})}-\hat{\mu}_{c}^{(\mathrm{G})}\right)^2\right]; \tag{36}$$

where Q_μ is the number of terms in the summand, subject to the constraints:

$$\kappa_{di}^c = \frac{\hat{\sigma}_{di,\mathrm{M}}^c - \hat{\sigma}_{d1,\mathrm{M}}^c}{\hat{\sigma}_{\mathrm{M,M}}^c}\left(\hat{\mu}_{\mathrm{M}}^{c(\mathrm{S})} - r_c\right) = \frac{\hat{\sigma}_{di,\mathrm{M}}^e - \hat{\sigma}_{d1,\mathrm{M}}^e}{\hat{\sigma}_{\mathrm{M,M}}^e}\left(\hat{\mu}_{\mathrm{M}}^{e(\mathrm{S})} - r_e\right) = \kappa_{di}^e \text{ (equation (35)).}$$

As explained above, the problem with this approach is that we may have more constraints than unknowns.

2.3.4 Instead we effect a compromise between the GCAPM and the SFM-CAPM. Instead of treating the SFM-CAPM condition as a strict constraint, we can treat it as part of the objective by minimising the penalty function:

$$D^2 = D_\mu^2 + hD_\kappa^2; \tag{37}$$

where:

$$D_\mu^2 = \frac{1}{Q_\mu}\left[\sum_{c=1}^{C}\left\{\sum_{i=2}^{n_c}\left(\hat{\mu}_{ci}^{(\mathrm{S})} - \hat{\mu}_{ci}^{(\mathrm{G})}\right)^2\right\} + \sum_{c=2}^{C}\left(\hat{\mu}_c^{(\mathrm{S})} - \hat{\mu}_c^{(\mathrm{G})}\right)^2\right] \text{ (equation (36));}$$

$$D_\kappa^2 = \frac{1}{Q_\kappa}\sum_{c,e=1}^{C}\sum_{(d,i)\in\Psi_c}\left(\kappa_{di}^c - \kappa_{di}^e\right)^2; \tag{38}$$

$$\kappa_{di}^f = \frac{\hat{\sigma}_{di,\mathrm{M}}^f - \hat{\sigma}_{d1,\mathrm{M}}^f}{\hat{\sigma}_{\mathrm{M,M}}^f}\left(\mu_{\mathrm{M}}^{f(\mathrm{S})} - r_f\right) \text{ (equation (20));} \tag{39}$$

Q_μ and Q_κ are the numbers of terms in the respective summands; and

h is a penalty coefficient.

2.3.5 This means that, whilst D_κ^2 will not generally be zero (which would be the case under the strict constraint) it can be reduced to an arbitrarily small value by increasing the penalty coefficient *h*. The estimates $\hat{\mu}_{ci}^{(\mathrm{S})}$ and $\hat{\mu}_c^{(\mathrm{S})}$ of the *ex-ante* underlying expectations will depend on *h*, as will the betas and the optimal portfolio. Bayesian credibility theory could be used to determine *h*. For $h=0$ the model reduces to the GCAPM as the constraints are not applied.

2.3.6 In terms of equation (39) κ_{di}^f is a function of μ_{M}^f. From equation (14) we have:

$$\mu_{\mathrm{M}}^f = \sum_{(c,j)\in\Psi_f} p_{cj}^f \mu_{cj}^f .$$

This means that κ_{di}^f is a function both of $\left\{p_{cj}^f \mid (c,j)\in\Psi_f\right\}$ and of $\left\{\mu_{cj}^f \mid (c,j)\in\Psi_f\right\}$.

2.3.7 Now each element of $\left\{p_{cj}^{f} \mid (c,j) \in \Psi_{f}\right\}$ is also a function of the underlying expectations. This is because the former, being a currency-*f* investor's optimal exposure to a particular asset, is dependent on the latter. It involves finding that investor's market portfolio as explained in section 2.1. What we therefore need to do is to find the values of the underlying expectations that minimise D^2.

2.3.8 First, for each currency *c*, the GCAPM underlying expectations $\hat{\mu}_{ci}^{(\mathrm{G})}$ and $\hat{\mu}_{c}^{(\mathrm{G})}$, the optimal market portfolios p_{cj}^{f} and the betas are determined for each investor currency. Using these underlying expectations as initial values, we then calculate the SFM-CAPM underlying expectations $\hat{\mu}_{ci}^{(\mathrm{S})}$ and $\hat{\mu}_{c}^{(\mathrm{S})}$—and hence, for a currency-*f* investor:

— the expected returns on the assets available $\left\{\hat{\mu}_{cj}^{f} \mid (c,j) \in \Psi_{f}\right\}$;

— the tangency portfolio $\left\{p_{cj}^{f} \mid (c,j) \in \Psi_{f}\right\}$;

— the expected returns to investors $\hat{\mu}_{\mathrm{M}}^{f(\mathrm{S})}$ on the tangency portfolio; and

— the generalised market risk premium κ_{di}^{f};

so as to minimise *D*. We can then also calculate the betas $\left\{\beta_{cj}^{f} \mid (c,j) \in \Psi_{f}\right\}$ such that:

$$\beta_{cj}^{f} = \frac{\sigma_{cj,\mathrm{M}}^{f}}{\sigma_{\mathrm{M,M}}^{f}}. \tag{40}$$

2.4 Local Optima

2.4.1 In applications of the method described above it was found that, for some values of *h*, the optimum value found by minimising D^2 was merely a local optimum, which did not conform to the theoretical requirements.

2.4.2 In the first place, the optimum value found should be independent of the initial value used for the iteration process followed in the optimisation function. It was found that this requirement was not invariably satisfied. For this reason, optimal values were found for a range of values of *h*. This range started with $h=0$, which gives the GCAPM global optimum, $\left(\hat{\mu}_{ci}^{(\mathrm{G})}, \hat{\mu}_{c}^{(\mathrm{G})}\right)$ being the *ex-post* sample values of the underlying expectations. For each subsequent value of *h* two values of $\left(\hat{\mu}_{ci}^{(h)}, \hat{\mu}_{c}^{(h)}\right)$—and hence of $\left(\left(D_{\mu}^{2}\right)^{(h)}, \left(D_{\kappa}^{2}\right)^{(h)}\right)$—were calculated, the first using $\left(\hat{\mu}_{ci}^{(\mathrm{G})}, \hat{\mu}_{c}^{(\mathrm{G})}\right)$ as initial values and the second using the SFM-CAPM values $\left(\hat{\mu}_{ci}^{(h-)}, \hat{\mu}_{c}^{(h-)}\right)$ found for *h*–, the previous value of *h*. The results that gave the lower value of D^2 were selected.

2.4.3 It was also found that, for different optimisation methods, different results were obtained. Again, this is due to different local optima. Again, the results that gave the lower value of D^2 were selected.

2.4.4 In theory, it may be shown that the locus of the solution in $D_\mu^2 - D_\kappa^2$ space should describe a monotonically decreasing function as h increases. For each pair of points $\left(\left(D_\mu^2\right)^{(h(1))},\left(D_\kappa^2\right)^{(h(1))}\right)$ and $\left(\left(D_\mu^2\right)^{(h(2))},\left(D_\kappa^2\right)^{(h(2))}\right)$ a check was made that they were monotonically decreasing. If that check failed, it was accepted that no global minimum of D^2 could be found for $h = h(2)$ and that value of h was ignored.

2.4.5 Not only should the locus of the solution in $D_\mu^2 - D_\kappa^2$ space describe a monotonically decreasing function, it should also describe a convex function. For each pair of points $\left(\left(D_\mu^2\right)^{(h(1))},\left(D_\kappa^2\right)^{(h(1))}\right)$ and $\left(\left(D_\mu^2\right)^{(h(3))},\left(D_\kappa^2\right)^{(h(3))}\right)$ a check was therefore made for the convexity of the values of $\left(\left(D_\mu^2\right)^{(h(1))},\left(D_\kappa^2\right)^{(h(1))}\right)$, $\left(\left(D_\mu^2\right)^{(h(2))},\left(D_\kappa^2\right)^{(h(2))}\right)$ and $\left(\left(D_\mu^2\right)^{(h(3))},\left(D_\kappa^2\right)^{(h(3))}\right)$ for $h(1) < h(2) < h(3)$. If that check failed, it was accepted that no global minimum of D^2 could be found for $h = h(2)$ and that value of h was ignored.

2.4.6 It is clear from the process described above that practitioners will not be able to predetermine a value of h and merely solve for that value. Instead, in order to avoid merely local optima, they will need to solve for a range of values of h. and then select a value of h that has not been rejected.

2.4.7 The resulting locus of the solution in $D_\mu^2 - D_\kappa^2$ space will describe a monotonically decreasing, convex function as h increases. Whilst there is no guarantee that the resulting values of D^2 will be global minima, they are not obviously merely local minima. Practitioners may wish to explore the possibility of lower minima using global optimisation methods, but a comprehensive discussion of the application of such methods is considered to be beyond the scope of this paper.

2.5 A Particular Case

If strict purchasing-power parity holds then, in real terms (or if there is no inflation), the SFM-CAPM reduces to the GCAPM. In this case:

$$\sigma_{c,\mathrm{M}}^c = 0;$$

and, for all currencies c and e:

$$\beta_{di}^{c} = \beta_{di}^{e}.$$

This means that

$$\kappa_{di}^{c} = \kappa_{di}^{e};$$

so:

$$D_{\kappa}^{2} = 0;$$

and the SFM-CAPM constraint applies trivially.

3. APPLICATION

For illustrative purposes the method outlined in sections 2.3 and 2.4 was applied to a selection of currencies. An overview of the data available is given in Thomson, Şahin & Reddy (op. cit.); it is not repeated here. However, for convenience, the datasets used are described in section 3.1. The results are presented in section 3.2.

3.1 Data

3.1.1 As explained in Thomson, Şahin & Reddy (op. cit.) it was decided to use various datasets for nominal returns and various datasets (not necessarily the same) for real returns. These datasets, comprising the periods, and the assets included in them, are shown in Table 1. In that table, 'e, cb' means equities and conventional bonds and 'ilb' means index-linked bonds.

Table 1 Periods used

Dataset	Period		USA		UK		SA		TR	
			e, cb	ilb	e, cb	ilb	e, cb	ilb	e, cb	ilb
Nominal returns										
1	1975Q2	1985Q4	√		√		√			
2	1986Q1	1995Q4	√		√		√		√	
3	1996Q1	2005Q2	√		√		√		√	
4	2005Q3	2012Q1	√		√		√		√	
5	2005Q3	2012Q1	√	√	√	√	√	√		
6	2009Q4	2012Q1	√	√					√	√
7	1975Q2	2012Q1	√		√		√			
8	1986Q1	2012Q1	√		√		√		√	
Real returns										
1	2003Q3	2009Q3	√	√	√	√				
2	2005Q3	2009Q3	√	√	√	√	√	√		
3	2009Q4	2012Q1	√	√					√	√
4	2005Q3	2012Q1	√	√	√	√	√	√		
5	2003Q3	2012Q1	√	√	√	√				

3.1.2 Some of the periods in Table 1 are too short for the estimation of reliable parameters; they are included for the sake of inclusivity and to indicate how they may affect the results. On the other hand, it must be recognised that means and covariances may change over time, so the use of excessively long periods is inappropriate. However, long periods have been included for the purposes of illustration.

3.2 Results

For nominal and real returns, and for each dataset listed in Table 1, optimal *ex-ante* expected values and the corresponding portfolios were determined as explained in section 2 and values of the GCAPM and SFM-CAPM betas were calculated, for a range of values of h. The results of these calculations are set out in this section.

3.2.1 ANALYSIS OF RESULTS IN D_μ–D_κ SPACE

3.2.1.1 As explained in section 2.3, the application of the SFM-CAPM requires the minimisation of $D^2 = D_\mu^2 + hD_\kappa^2$ (equation (37)). For $h=0$ this reduces to the minimisation of D_μ^2 (equation (36)). In fact, with no constraints, we obtain $D_\mu^2 = 0$, so that $\hat{\mu}_{ci}^{(\mathrm{S})} = \hat{\mu}_{ci}^{(\mathrm{G})}$ and $\hat{\mu}_{c}^{(\mathrm{S})} = \hat{\mu}_{c}^{(\mathrm{G})}$ for all (c, i) and the SFM-CAPM reduces to the GCAPM. As h increases, the SFM-CAPM departs from the GCAPM; D_μ^2 increases and D_κ^2 decreases.

3.2.1.2 Figures 1 and 2 show, for nominal and real returns respectively, and for the datasets as enumerated in the legend, the values of D_μ and D_κ for optimal values of D^2. On the horizontal axis D_μ shows the root-mean-square average of the differences between the underlying expectations under the SFM-CAPM and those under the GCAPM. On the vertical axis D_κ shows the root-mean-square average of the differences of the generalised market risk premiums between currencies. The intercept on the vertical axis shows the value of D_κ for $D_\mu = 0$; i.e. the GCAPM value of D_κ. For the sake of comparability the same scale has been used in both figures.

3.2.1.3 As explained in section 2.4, the locus of $\left(\left(D_\mu^2\right)^{(h)}, \left(D_\kappa^2\right)^{(h)}\right)$ describes a monotonically decreasing function as h increases. In Figure 1 it may be observed that, for dataset 4 for example, the locus is not convex. This is because, whilst the figure is presented in $D_\mu^{(h)} - D_\kappa^{(h)}$ space, convexity is required in $\left(D_\mu^{(h)}\right)^2 - \left(D_\kappa^{(h)}\right)^2$ space, since the objective function is expressed in terms of the latter; in that space all the loci are convex. Because of the rejection of local optima, some of the lines between one value and the next are quite long. (The lines themselves do not represent valid values; they merely connect the points at which successive values of h produce accepted results. These connections are important because they show that the locus decreases monotonically as h increases.)

3.2.1.4 Figure 1 shows that, for nominal dataset 3, the value of D_κ for $h=0$ (i.e. for the GCAPM) is 0.019. This means that the GCAPM results are not very close to the SFM-CAPM results. Nevertheless, it decreases rapidly as D_μ increases. For all the datasets D_κ reaches below 0.004, but in no case was it possible to find values below 0.0004; the lowest values ranged across datasets from 0.0004 to 0.004. In practice it is not possible to obtain arbitrarily low values of D_κ; eventually D_μ—and therefore the level of the underlying expectations—becomes irrelevant.

3.2.1.5 Figure 2 shows the corresponding results for real returns. The values of D_κ for datasets 2 and 4 for $h=0$ are particularly high. Nevertheless, for all the datasets D_κ again reaches below 0.004, but in no case was it possible to find values below 0.0005.

3.2.2 UNDERLYING EXPECTATIONS

3.2.2.1 Figure 3 shows, for nominal returns, the values of the SFM-CAPM underlying expectations $\hat{\mu}_{di}^{(S)}$ and $\hat{\mu}_{d}^{(S)}$ against the GCAPM underlying expectations $\hat{\mu}_{di}^{(G)}$ and $\hat{\mu}_{d}^{(G)}$. For the purposes of this and subsequent figures, values of h have been selected from the results so as to show the effects of different penalty coefficients on the optimal values of the variables concerned. For nominal returns the values were selected from the results shown in Figure 1 to give $D_\mu \approx 0.002, 0.005, 0.008$, representing low, medium and high levels of departure of the *ex-ante* estimates of the expected values from the *ex-post* estimates, i.e. low, medium and high levels of credibility of the SFM-CAPM. In the legend these values are referred to as 'low D.mu', 'medium D.mu' and 'high D.mu'

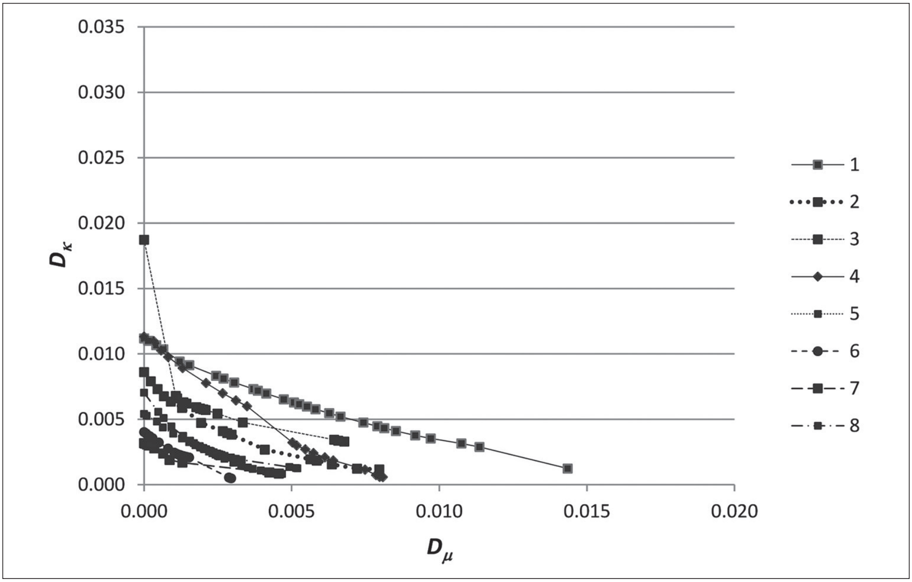

FIGURE 1 D_μ and D_κ for optimal values of D^2: nominal returns respectively. They may be compared with the line referred to as 'zero D.mu', which represents the GCAPM value.

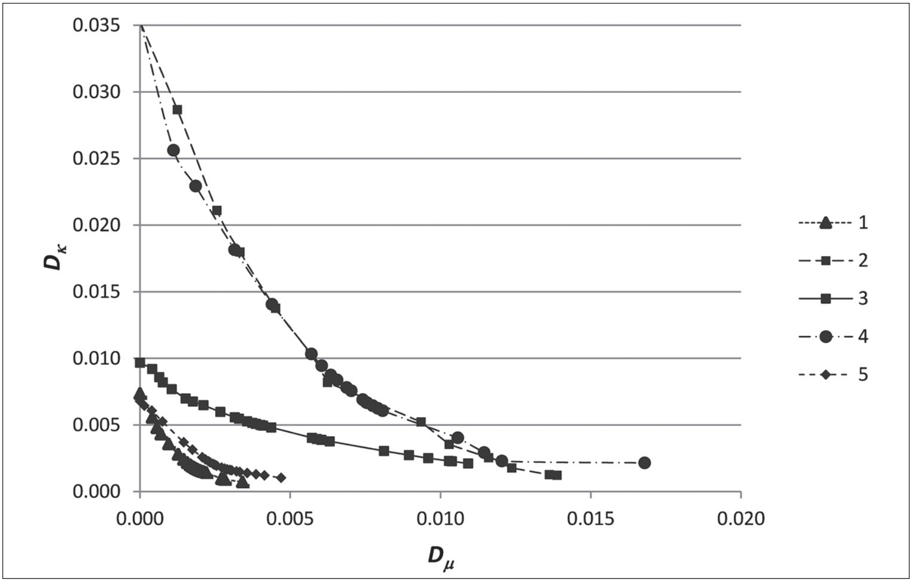

FIGURE 2 D_μ and D_κ for optimal values of D^2: real returns

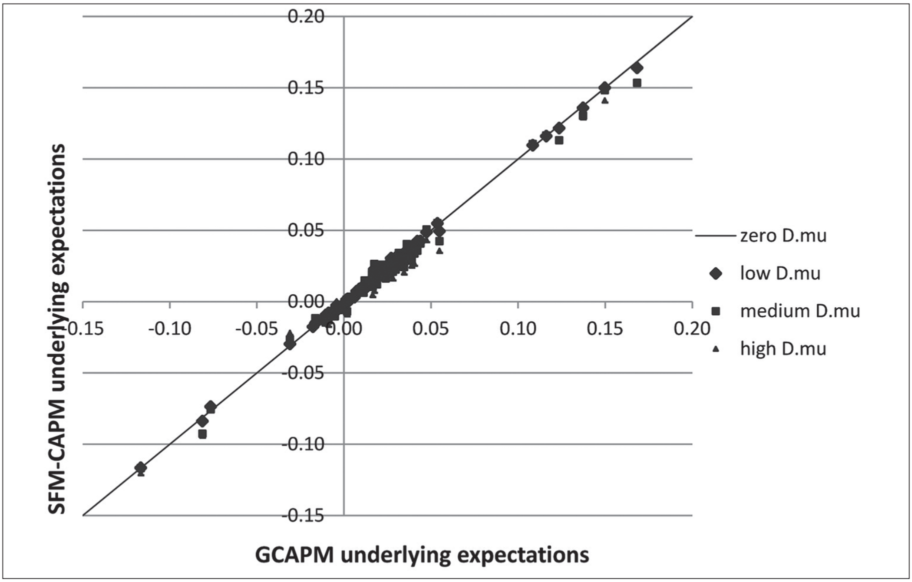

FIGURE 3 Underlying expectations: nominal returns

3.2.2.2 As observed in the preceding section, not all datasets have values of D_μ reaching 0.008, and medium and high levels are omitted where necessary.

3.2.2.3 The SFM-CAPM values are clustered around the GCAPM (zero D.mu) line. Nevertheless, in relation to investment-management decision-making the differences are material. As might be expected, the relatively few values for $D_\mu \approx 0.008$ (high D_μ) frequently appear outside of the rest. The values are more-or-less clustered into three groups: the higher cluster is for expected returns on assets issued in Turkish lira, whilst the lower cluster is for negative expected strengthening of the Turkish lira. The high nominal returns on Turkish assets (both bonds and equities) are offset by the expected weakening of that currency.

3.2.2.4 Because of the large number of points in Figure 3, some of the detail is lost. By way of illustration, Figure 4 gives the same information for nominal-returns dataset 3 only. In that figure, for the sake of clarity, the outlying clusters have been omitted and a larger scale has been used. Here it may be seen that, for each value of GCAPM underlying expectations, there are three values of SFM-CAPM underlying expectations, the low D_μ value being the closest to the GCAPM and the high D_μ value the furthest.

3.2.2.5 Figure 5 gives, for real returns, information corresponding to Figure 3. For the sake of comparability the same scale has been used. For nominal returns the

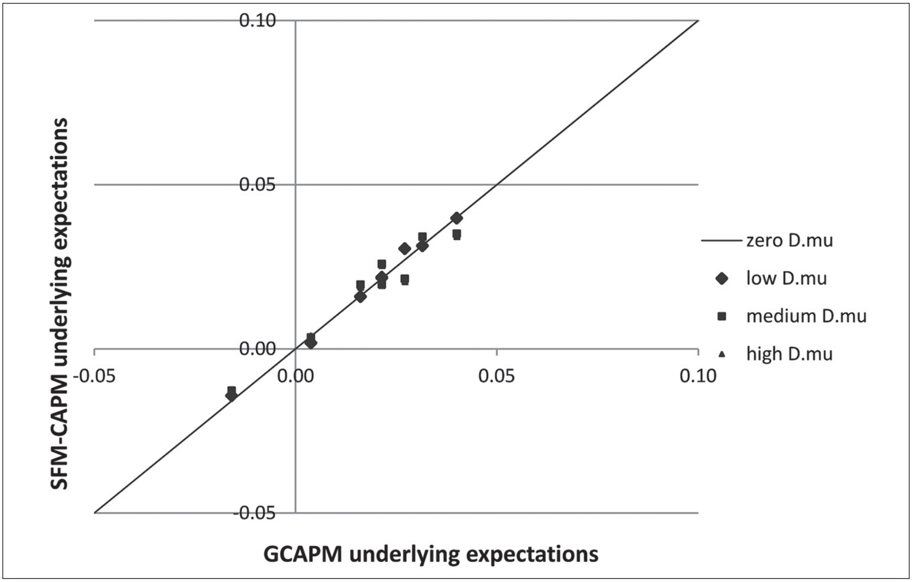

Figure 4 Underlying expectations: nominal returns dataset 3

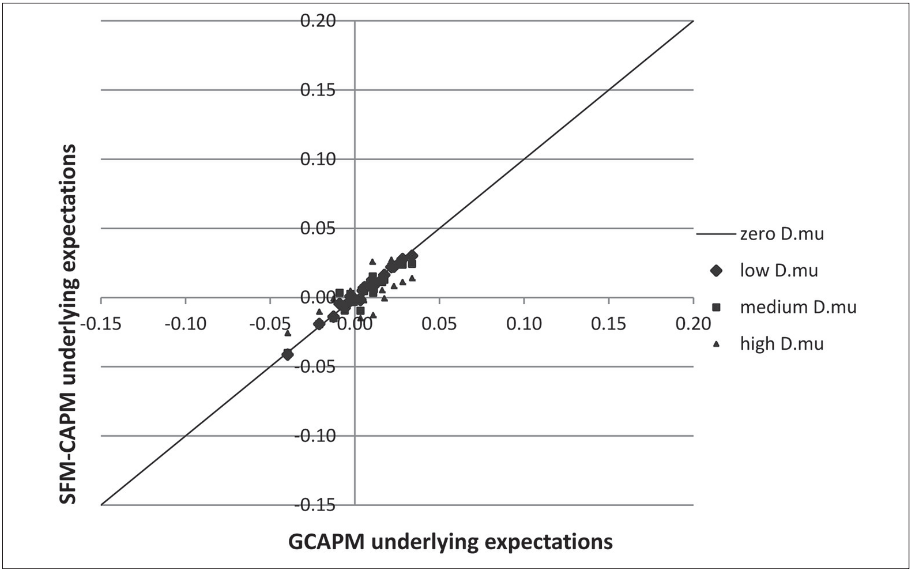

FIGURE 5 Underlying expectations: real returns

values were selected from the results shown in Figure 1 to give $D_\mu \approx 0.002, 0.005, 0.012$, representing low, medium and high levels of departure of the *ex-ante* estimates of the expected values from the *ex-post* estimates, i.e. low, medium and high levels of credibility of the SFM-CAPM. As for nominal returns, these values are referred to as 'low D.mu', 'medium D.mu' and 'high D.mu' respectively. Here the high and low clusters do not occur; high expected returns on Turkish assets and expected weakening of the Turkish lira are offset by high inflation. Again the SFM-CAPM values are clustered around the GCAPM (zero D.mu) line. Nevertheless, in relation to investment-management decision-making the differences are material for high D_μ. For high values of h (and therefore of D_μ) underlying expectations under the SFM-CAPM may be negative, even for positive GCAPM values.

3.2.3 EXPECTED RETURNS TO INVESTORS

3.2.3.1 Figure 6 shows, for nominal returns, the values of $\hat{\mu}^c_{di}$, i.e. the expected returns to investors. In comparison with Figure 3, two features are noteworthy: first, the SFM-CAPM values are more dispersed around the GCAPM values and secondly, the negative SFM-CAPM values are substantially less material than those of underlying expectations. This is because:

$$\mu^c_{di} = \mu_{di} + \mu_d - \mu_c \quad \text{(equation (6))}.$$

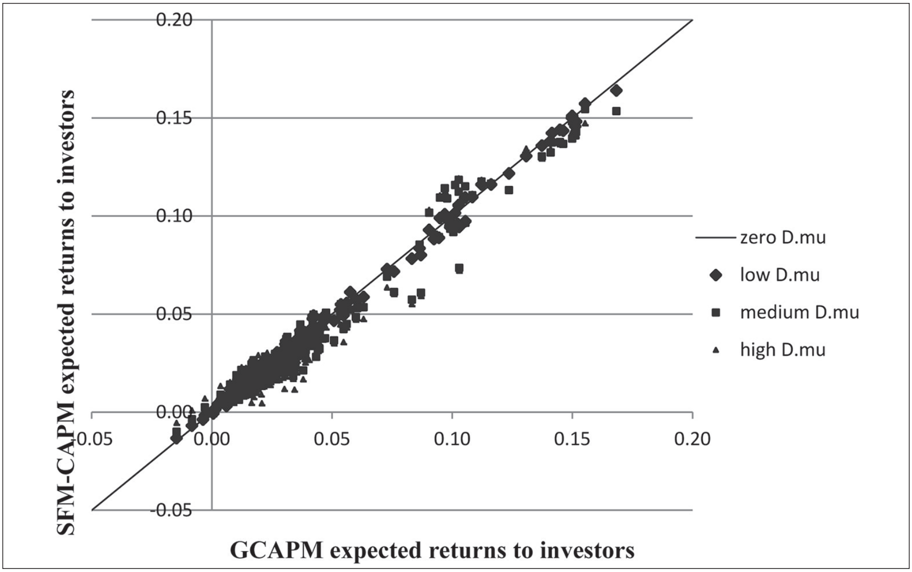

FIGURE 6 Expected returns to investors: nominal returns

3.2.3.2 Firstly, to the extent that μ_c and μ_d vary independently of each other, the variability of μ_{di}^c will be greater than that of μ_{di}. And secondly, if $\mu_d - \mu_c >> 0$ then $\mu_{di}^c >> \mu_{di}$ and vice versa.

3.2.3.3 Figure 7 gives, for real returns, information corresponding to Figure 6. Here the negative values are more evident. This is because, being net of inflation, they are generally lower. Also, the range of values is lower. This is because, when expected returns are high, they may to a large extent be offset by high expected rates of inflation. Here a wider spread of values for high D_μ is noticeable.

3.2.4 OPTIMAL PORTFOLIOS

3.2.4.1 Figures 8 to 10 show, for nominal returns, the values of $p_{di}^{c(S)}$, i.e. the optimal portfolio weightings of a currency-*c* investor in asset *i* issued in currency *d* under the SFM-CAPM in comparison with those under the GCAPM. Figure 8 shows investments in the investor's home currency, Figure 9 shows investments by USA and UK investors in assets issued in South Africa and Turkey. Figure 10 shows other investments.

3.2.4.2 There are numerous points at the origin representing zero exposure under both models. Otherwise there is not much consistency between the GCAPM portfolios and the SFM-CAPM portfolios, even for low D_μ; the portfolios are very sensitive to deviations in the underlying expectations from those of the GCAPM.

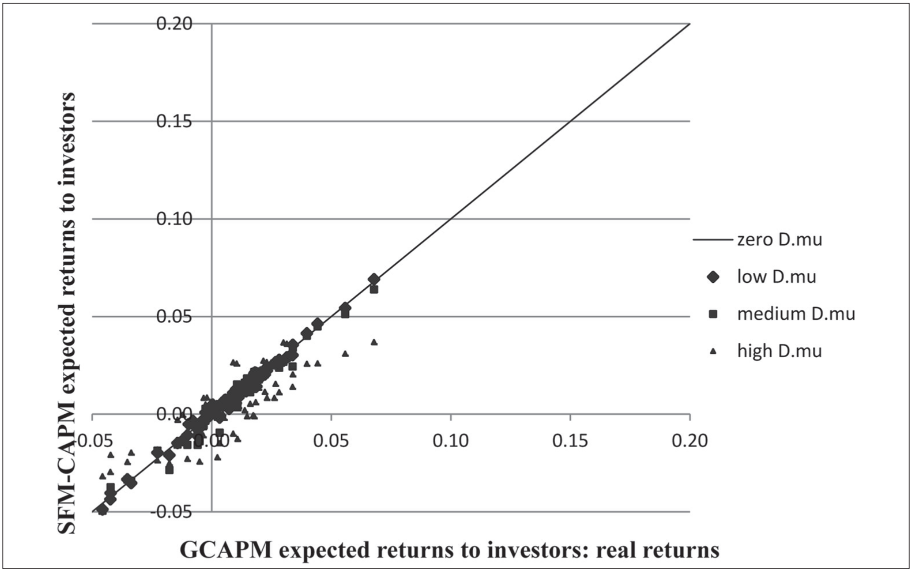

FIGURE 7 Expected returns to investors: real returns

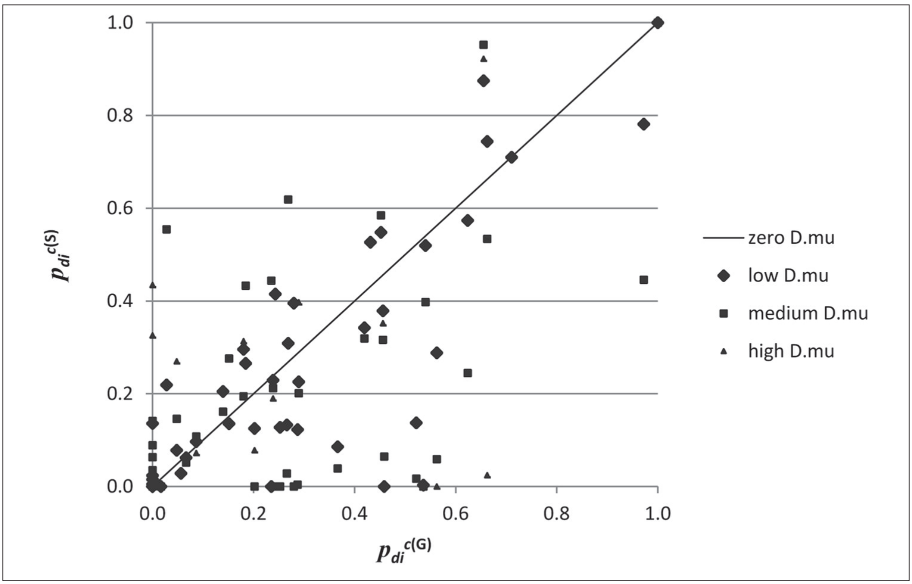

FIGURE 8 $p_{di}^{c(S)}$: nominal returns: investments in home currency

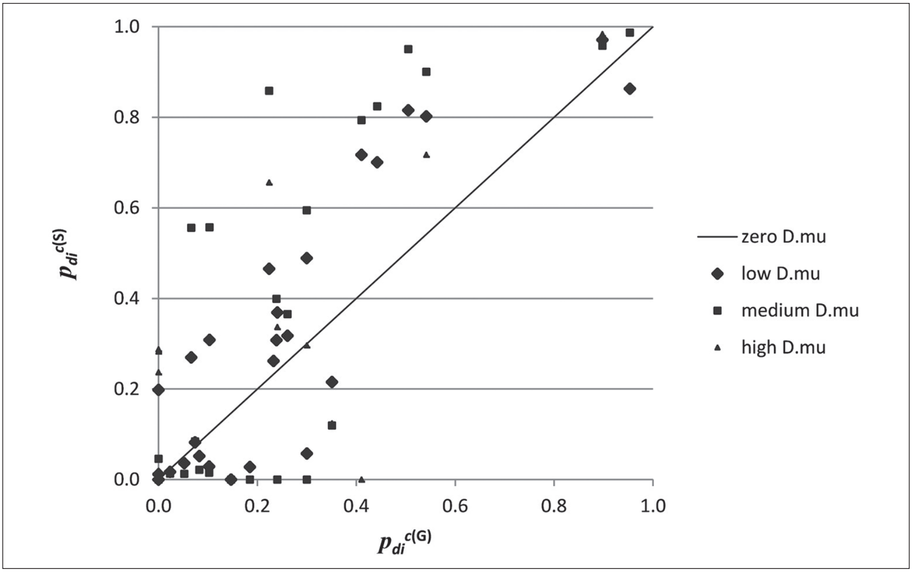

FIGURE 9 $p_{di}^{c(S)}$: nominal returns: investments in smaller currencies

3.2.4.3 The sheer size of the major currencies makes it impossible that substantial proportions of their investments are in assets of minor currencies. It should be recognised that the optimisation of the portfolio choice makes no allowance for home bias. However, Figure 9 shows that the exposure of USA and UK investors to smaller currencies tends to be higher under the SFM-CAPM than under the GCAPM. A higher proportion of the points are at the origin than for home investment. These results suggest that home bias is better justified under the GCAPM than under the SFM-CAPM. Some of the points represent exposures that would be unattainable in practice. In practice, it would be necessary to limit exposures to attainable proportions. The results of this research must be qualified by the understanding that they represented the results that would have obtained if the major-currency investors could have invested substantial proportions of their assets in minor currencies. Further research is required calibrating *ex-ante* underlying expectations to market portfolios both under the GCAPM and under the SFM-CAPM.

3.2.4.4 Figure 10 shows similar tendencies to Figure 9.

3.2.4.5 Figures 11 to 13 show corresponding results for real returns. As shown in Figures 12 and 13, the number of investments in smaller currencies and in other foreign currencies is relatively low, both under the GCAPM and under the SFM-CAPM. In both cases the level of exposure is also relatively low. This suggests that both models better justify home bias for real returns than for nominal returns. The adjustments required by calibration to actual exposures are therefore likely to be less substantial.

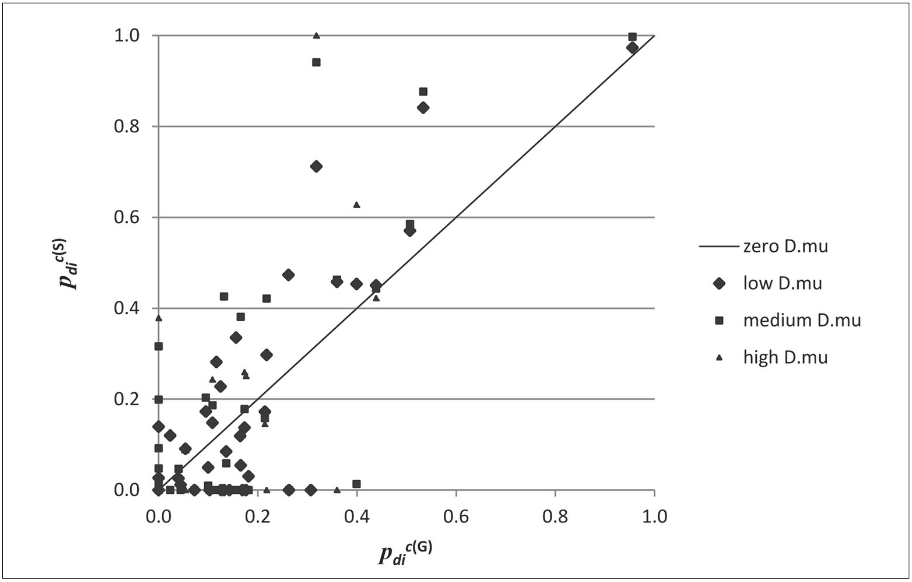

FIGURE 10 $p_{di}^{c(S)}$: nominal returns: investments in other currencies

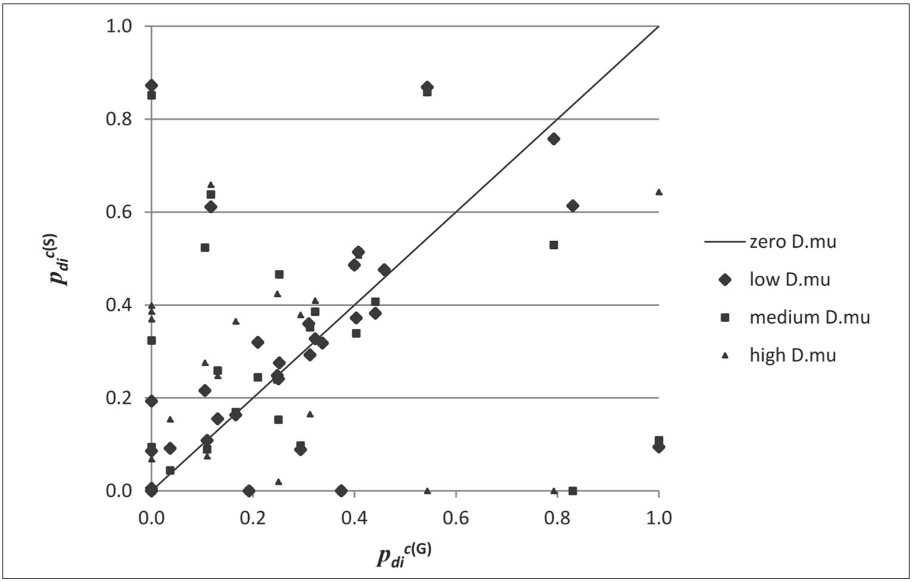

FIGURE 11 $p_{di}^{c(S)}$: real returns: investments in home currency

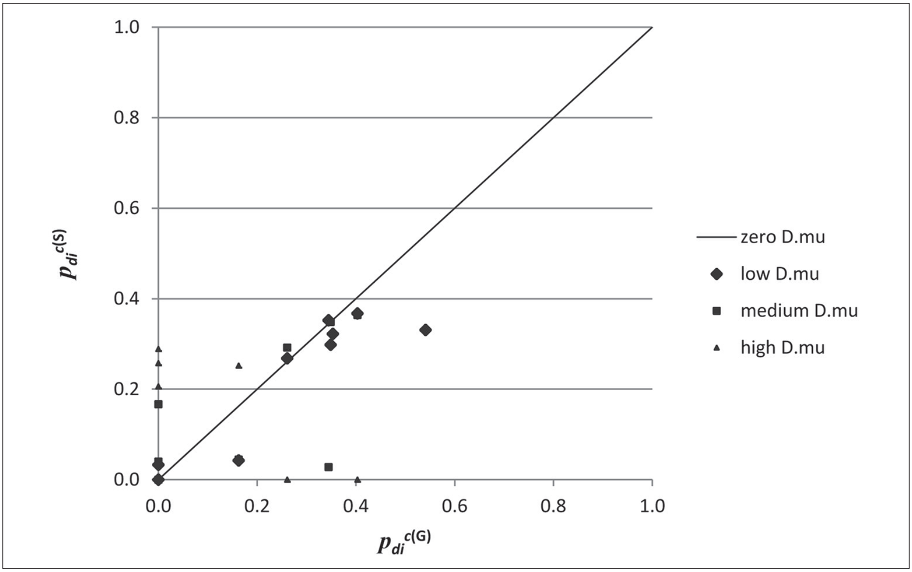

FIGURE 12 $p_{di}^{c(S)}$: real returns: investments in smaller currencies

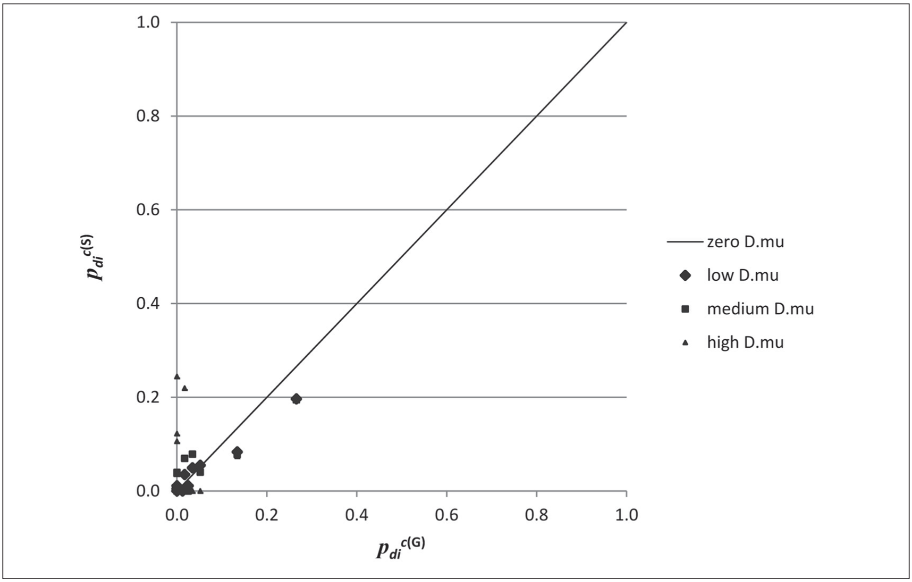

FIGURE 13 $p_{di}^{c(S)}$: real returns: investments in other currencies

3.2.5 BETA

3.2.5.1 It was expected that the SFM-CAPM estimates of beta would be approximately equal to the GCAPM estimates, with some shift to represent the effects of the SFM-CAPM requirements and some noise. An upward shift would indicate that the sample betas are understated and a downward shift would indicate that they are overstated. The shifts indicate the corrections required to the sample betas by the underlying assumptions of the SFM-CAPM.

3.2.5.2 Figure 14 shows the relationship of the SFM-CAPM estimates of the betas to the GCAPM estimates for nominal returns. The bulk of the SFM-CAPM estimates are clustered about the GCAPM (zero D.mu) line, but there is a tendency for the former to be lower than the latter, especially at relatively high values and relatively low values. Even for intermediate values the tendency is noticeable.

3.2.5.3 Figure 15 shows the relationship of the SFM-CAPM betas to the GCAPM betas for real returns. There are numerous unrealistically high values, both under the GCAPM and under the SFM-CAPM. Most of the high values were for USA long conventional bonds. For high D_μ, though, the SFM-CAPM tends to show more realistic results than the GCAPM. It should be recognised that the assumption that the *ex-ante* betas are equal to the *ex-post* betas may result in anomalies. For an asset whose returns show a variance that is much higher than that of the returns on the market portfolio but are strongly correlated with the latter, the *ex-post* betas will be high. This

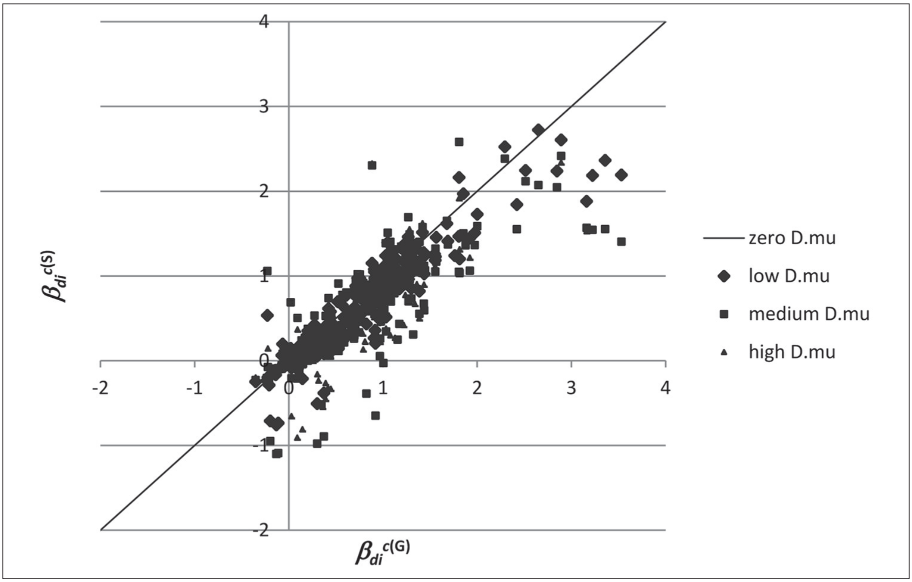

Figure 14 SFM-CAPM beta versus GCAPM beta: nominal returns

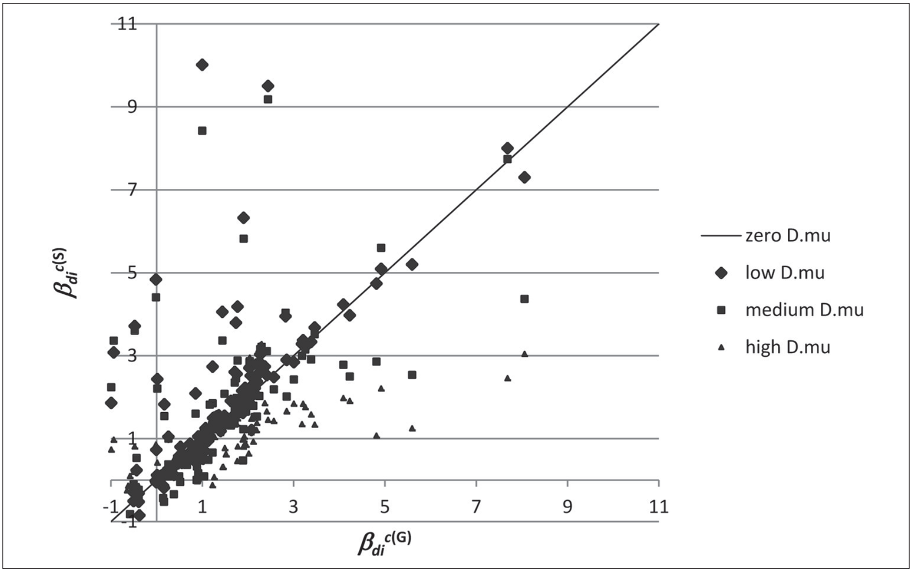

FIGURE 15 SFM-CAPM beta versus GCAPM beta: real returns

may be exacerbated by individual years in which both the return on the asset and the return on the market portfolio are high relative to their means, but the former is much higher than the latter. It may also be exacerbated by individual years in which both the return on the asset and the return on the market portfolio are low relative to their means, but the former is much lower than the latter. Where there are many assets in the opportunity set, this may happen quite fortuitously, either in the GCAPM or in the SFM-CAPM. Under both models care needs to be taken in the estimation of the betas; outliers should be disregarded.

4. CONCLUSIONS

4.1 Summary

4.1.1 It is shown above that, for a single-factor CAPM to work in a multi-currency world, there is a necessary condition. That condition applies to the *ex-ante* variances and covariances of returns. The resulting SFM-CAPM model developed in this paper may be specified as:

$$\mu_{di}^{c} = r_c + \beta_{di}^{c}\left(\mu_{\mathrm{M}}^{c} - r_c\right);$$

where:

$$\mu_{\mathrm{M}}^{c} = E\left\{X_{\mathrm{M}}^{c}\right\} = \sum_{(d,i)\in\Psi_c} p_{di}^{c}\mu_{di}^{c}$$

$X_{\mathrm{M}}^{c} =$ is the return in currency c on the tangency portfolio of a currency-c investor;

$\mu_{di}^{c} = E\left\{X_{di}^{c}\right\}$;

$X_{di}^{c} = X_{di} + X_{d} - X_{c}$ is the return in currency *c* on asset *i* issued in currency *d* for $c = 1,\ldots,C;\ (d,i) \in \Psi_{c}$;

X_{di} is the return in currency *d* on asset *i* issued in that currency for $d = 1,\ldots,C;\ (d,i) \in \Psi_{d}$, so that $X_{di} = X_{di}^{d}$, $X_{d1} = r_{d}$;

X_{e} is the rate of strengthening of currency *e* relative to an arbitrarily chosen currency 1;

$\Psi_{c} = \left\{(d,i) \mid d \in \left\{1,\ldots,C\right\}; i \in \Omega_{d}^{c}\right\}$;

$$\Omega_{d}^{c} = \begin{cases} \left\{2,\ldots,n_{d}\right\} \text{ for } d = c; \\ \left\{1,\ldots,n_{d}\right\} \text{ for } d \neq c; \end{cases}$$

$i = 1$ denotes the risk-free asset in currency *d* and $i > 1$ a risky asset;

r_{c} is the return on the risk-free asset denominated in currency *c*;

$\sigma_{di,\mathrm{M}}^{c} = \mathrm{cov}\left\{X_{di}^{c}, X_{\mathrm{M}}^{c}\right\} = \sum_{(e,j)\in\Psi_{c}} p_{ej}^{c}\sigma_{di,ej}^{c}$;

$\sigma_{\mathrm{M,M}}^{c} = \mathrm{var}\left\{X_{\mathrm{M}}^{c}\right\} = \sum_{(d,i),(e,j)\in\Psi_{c}} p_{di}^{c}p_{ej}^{c}\sigma_{di,ej}^{c}$;

$\sigma_{di,ej}^{c} = \mathrm{cov}\left\{X_{di}^{c}, X_{\mathrm{ej}}^{c}\right\}$; and

$\left\{p_{cj}^{f} \mid (c,j) \in \Psi_{f}\right\}$ is the tangency portfolio of a currency-*c* investor, which maximises:

$$k = \frac{\mu_{\mathrm{M}}^{c} - r_{c}}{\sqrt{\sigma_{\mathrm{M,M}}^{c}}};$$

subject to the constraints:

$p_{di}^{c} \geq 0$ for all $\left(d,i\right) \in \Psi_{c}$ and for all *c*; and

$$\sum_{(d,i)\in\Psi_{c}} p_{di}^{c} = 1.$$

All returns on assets and rates of strengthening of currencies are expressed as forces.

4.1.2 In practice this model generally has more constraints than unknowns, so the constraints cannot be applied rigorously. However, the condition may be applied by means of a penalty method by finding $\hat{\mu}_{ci}^{(\mathrm{S})}$ and $\hat{\mu}_{c}^{(\mathrm{S})}$ so as to minimise:

$$D^2 = D_\mu^2 + hD_\kappa^2;$$

where:

$$D_\mu^2 = \frac{1}{Q_\mu}\left[\sum_{c=1}^{C}\left\{\sum_{i=2}^{n_c}\left(\hat{\mu}_{ci}^{(\mathrm{S})} - \hat{\mu}_{ci}^{(\mathrm{G})}\right)^2\right\} + \sum_{c=2}^{C}\left(\hat{\mu}_{c}^{(\mathrm{S})} - \hat{\mu}_{c}^{(\mathrm{G})}\right)^2\right];$$

$$D_\kappa^2 = \frac{1}{Q_\kappa}\sum_{c,e=1}^{C}\sum_{(d,i)\in\Psi_c}\left(\kappa_{di}^{c} - \kappa_{di}^{e}\right)^2;$$

$$\kappa_{di}^{f} = \frac{\hat{\sigma}_{di,\mathrm{M}}^{f} - \hat{\sigma}_{d1,\mathrm{M}}^{f}}{\hat{\sigma}_{\mathrm{M,M}}^{f}}\left(\hat{\mu}_{\mathrm{M}}^{f(\mathrm{S})} - r_f\right);$$

Q_μ and Q_κ are the numbers of terms in the respective summands; and

h is a penalty coefficient.

4.1.3 In the definition of D_μ^2, the superscripts (S) and (G) refer to the SFM-CAPM and the GCAPM respectively. Under the GCAPM, the underlying expectations $\hat{\mu}_{ci}^{(\mathrm{G})}$ and $\hat{\mu}_{c}^{(\mathrm{G})}$ are the *ex-post* sample means of X_{ci} and X_c respectively.

4.1.4 This means that, whilst D_κ^2 will not generally be zero, it can theoretically be reduced to an arbitrarily small value by increasing the penalty coefficient h. However, in practice, it is not possible to obtain an arbitrarily small value of D_κ^2. The estimates $\hat{\mu}_{ci}^{(\mathrm{S})}$ and $\hat{\mu}_{c}^{(\mathrm{S})}$ of the *ex-ante* underlying expectations will depend on h, as will the betas and the optimal portfolio. Bayesian credibility theory could be used to determine h. Otherwise it is a matter to which professional judgement should be applied.

4.1.5 The resulting SFM-CAPM developed in this paper may be applied as:

$$\hat{\mu}_{di}^{c} = r_c + \beta_{di}^{c}\left(\hat{\mu}_{\mathrm{M}}^{c} - r_c\right);$$

where:

$$\beta_{di}^{c} = \frac{\hat{\sigma}_{di,\mathrm{M}}^{c}}{\hat{\sigma}_{\mathrm{M,M}}^{c}}.$$

4.1.6 The model was applied to major categories of assets issued in the USA, the UK, South Africa and Turkey.

4.1.7 Where the initial value of the SFM-CAPM D_{κ} for $h=0$ was high, it was possible to reduce it to less than 0.004. In none of the datasets considered did the value decrease below 0.0004.

4.1.8 The SFM-CAPM values of underlying expectations are quite close to the GCAPM values, particularly for real returns. Nevertheless, in relation to investment-management decision-making the differences are material. When expected returns to investors are considered the differences between the GCAPM and the SFM-CAPM become more substantial.

4.1.9 With regard to optimal portfolios there was not much consistency between the GCAPM portfolios and the SFM-CAPM portfolios, even for low D.mu; the portfolios were very sensitive to deviations in the underlying expectations from those of the GCAPM. The results suggest that home bias is better justified for real returns than for nominal returns. For nominal returns it is better justified under the GCAPM than under the SFM-CAPM. Some of the optimal exposures would be unattainable in practice. The results must be interpreted as those that would have obtained if the major-currency investors could have invested substantial proportions of their assets in minor currencies. In practice, it would be necessary to limit exposures to attainable proportions. Because home bias is better justified for real returns, this may be easier for real returns than for nominal returns.

4.1.10 For nominal returns, the bulk of the SFM-CAPM estimates are clustered about the GCAPM line, but there is a tendency for the former to be lower than the latter, especially at relatively high values and relatively low values. Even for intermediate values the tendency is noticeable. For real returns there are numerous unrealistically high values, both under the GCAPM and under the SFM-CAPM. For high D_{μ}, though, the SFM-CAPM tends to shows more realistic results than the GCAPM. Under both models care needs to be taken in the estimation of the betas.

4.1.11 The findings of this paper give adequate grounds for the implementation of the SFM-CAPM. It is preferable to multi-factor models in that it does not treat currency risks as carrying different weight from investment risks; regardless of its source, risk is measured as variance in returns in the investor's currency and weighted accordingly. It is preferable to the GCAPM in that the implied price of a security to an investor who measures returns in a particular currency is the same as the price to an investor who measures returns in another currency. As shown in this paper, the results produced by the SFM-CAPM become materially different from those of the GCAPM as h increases.

4.1.12 The paper suggests that, if this model is to be applied, it would be better to apply it to real returns than to nominal returns. As noted in ¶2.1.5 above, practitioners commonly record and report returns in nominal terms. The use of real returns may therefore present some challenges However, it needs to be borne in mind that the CAPM derives its validity from the optimisation of consumption, which is expressed in real terms rather than in nominal terms. Furthermore, to the extent that a client's liabilities are expressed in real terms, an analysis of investment returns in real terms is more appropriate. Nevertheless, even in nominal terms, where the portfolios produced by the optimisation procedure are credible and the market risk premiums are realistic, the results of this paper show that the SFM-CAPM produces reasonable results.

4.2 Further Research

4.2.1 In practice the stochastic modelling of the assets and liabilities of a long-term financial institution requires the use of time series in which the expected returns on assets and the expected forces of inflation, and perhaps their variances and covariances, may vary over time. This means that the application of the SFM-CAPM to such modelling will necessitate the use of the method in a time series. For that purpose the time-series model may be used to simulate, at the start of each year, estimates of expected returns on the market portfolio during the forthcoming year and of the variances and covariances (for example in terms of a GARCH model)—and hence the betas—of each asset category. The SFM-CAPM may then be used to estimate expected returns during that year, conditional on the simulated estimates, of the returns on each asset category. These estimates may then be used to simulate returns on each asset category. For the SFM-CAPM the distribution of the return on the market portfolio and on each asset category may (as for other versions of the CAPM) be taken as any elliptically symmetric distribution. In the single-currency case the theory of this process has been developed in Thomson and Gott (2009) and an application has been demonstrated in Thomson (2011). Its application in the multi-currency case awaits further research. Because the SFM-CAPM is an equilibrium model, it is well suited to such applications. The advantage of equilibrium models for such purposes is that they do not assume that the investor can outperform the market on a risk-adjusted basis, thus allowing market consistency, and that, unlike more general no-arbitrage models, they do not assume complete markets, thus allowing for the fact that the liabilities of a financial institution cannot be replicated in the market. Whilst the assumption of equilibrium is inappropriate for an investment manager whose mandate is to outperform a benchmark portfolio, it is appropriate for the formulation of such a portfolio, and the SFM-CAPM may be used for such purposes.

4.2.2 The calibration of underlying expectations to the sizes of markets and of the amounts of the various asset categories available is a matter for further research. Alternatively, home bias and constraints on investment abroad can be allowed for by limitations to the amounts of assets in which foreign investors will invest as contemplated in ¶2.1.27. Some combination of these approaches may be appropriate.

4.2.3 In reporting to clients and in the application of the SFM-CAPM to the stochastic modelling of the assets and liabilities of a financial institution, practitioners will have to allow for those effects. This may be done by using equation (35A) to determine sample values of the residual:

$$\varepsilon_{di}^{c} = \left(X_{di}^{c} - r \right)_{c} - \beta_{di}^{c} \left(X_{\mathrm{M}}^{c} - r_{c} \right).$$

It may be instructive to investigate the historical performance of that residual.

4.2.4 It may be of interest to explore the possibility of relaxing assumption (2) by allowing, for example, for more than one set of investors in currency *c*, each set having its own expectations: homogeneous within the set but heterogeneous between sets.

4.2.5 The principal interest of the authors is in the development of stochastic models for actuarial use. Nevertheless, the SFM-CAPM clearly does have wider application—for example in determining cost of capital. For such applications it is not necessarily envisaged that this model will replace other models, but subject to the results of the further research suggested here, there is no reason why the SFM-CAPM should not take its place alongside other models in informing subjective assessment by practitioners of the expected returns on assets in a multi-currency world.

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